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EIGENFUNCTION ANALYSIS FOR BENDING OF CLAMPED RECTANGULAR, ORTHOTROPIC PLATES

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Abstract—Eigenfunctions have been developed for the clamped, rectangular orthotropic plate undergoing small deflections. The familiar Fadle–Papkovitch [1] functions for the isotropic case are shown to form a special case of the "orthotropic eigenfunctions". An analysis for orthotropic plates using these functions yields extremely accurate solutions.

1. INTRODUCTION

ONE of the useful methods for problems of bending of clamped, rectangular isotropic plates is the method of eigenfunctions. The literature on this is extensive; see [1–9]. Tentative solutions of the biharmonic equation are postulated and a transcendental eigenvalue equation is obtained by satisfying homogeneous boundary conditions at the clamped edges. Solution of the eigenvalue equation gives rise to expansions in terms of eigenfunctions, a linear combination of which is generally sufficient to satisfy the boundary conditions on the remaining edges.

The purpose of this work is to develop eigenfunctions for the clamped, rectangular, orthotropic plate undergoing "small" deflections, and to present an analysis of the problem using these functions. These functions have not been developed before; the Fadle–Papkovitch [1, 2] functions, well-known for the isotropic case, form a special case of the functions developed here. The clamped orthotropic plate problem can, of course, be treated by other more familiar techniques discussed in Timoshenko [10] and Lechnitzky [11]. The present analysis with the eigenfunctions yields an extremely accurate solution and is well suited for machine computation.

1.1. *Statement of the problem*

We consider a clamped, rectangular, orthotropic plate of size $a \times b$ and thickness, h. The material has three planes of symmetry with respect to its elastic properties. Following Timoshenko [10] we choose these planes as the co-ordinate planes (Fig. 1) and assume the

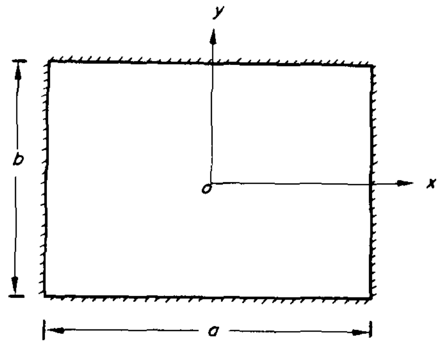

FIG. 1. The clamped, rectangular, orthotropic plate.

stress–strain relations for the case of plane stress in x–y plane as:

$$\begin{aligned}\sigma_x &= E'_x\varepsilon_x + E''\varepsilon_y \\ \sigma_y &= E'_y\varepsilon_y + E''\varepsilon_x \\ \tau_{xy} &= G\gamma_{xy}.\end{aligned} \tag{1}$$

The governing differential equation for the deflection, w, of the plate under transverse load Q, is

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = Q(x, y) \tag{2}$$

where

$$D_x = E'_x h^3/12; \quad D_y = E'_y h^3/12; \quad H = D_1 + 2D_{xy}$$
$$D_1 = E''h^3/12; \quad D_{xy} = Gh^3/12.$$

The bending moments are given by

$$\begin{aligned}M_x &= -\left(D_x\frac{\partial^2 w}{\partial x^2} + D_1\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -\left(D_y\frac{\partial^2 w}{\partial y^2} + D_1\frac{\partial^2 w}{\partial x^2}\right).\end{aligned} \tag{3}$$

The boundary conditions are

$$w = \frac{\partial w}{\partial y} = 0 \quad \text{on } y = \pm b/2 \tag{4.1}$$

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{on } x = \pm a/2. \tag{4.2}$$

2. EIGENFUNCTIONS

Consider the homogeneous equation

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = 0. \tag{5}$$

Assume tentatively a solution in the form

$$w = \cosh\left(\frac{2\rho x}{b}\right)F(y)$$

and let

$$w = \frac{\partial w}{\partial y} = 0 \text{ at } y = \pm b/2.$$

F is then determined by

$$\frac{d^4F}{dy^4}+2C_1\left(\frac{4\rho^2}{b^2}\right)\frac{d^2F}{dy^2}+C_2^2\left(\frac{16\rho^4}{b^4}\right)F = 0 \tag{6}$$

under the conditions $F = dF/dy = 0$ at $y = \pm b/2$.

The constants C_1 and C_2 in equation (6) are given by

$$\begin{aligned} C_1 &= H/D_y = (D_1+2D_{xy})/D_y = (E''+2G)/E'_y \\ C_2 &= \sqrt{(D_x/D_y} = \sqrt{(E'_x/E'_y)}. \end{aligned} \tag{7}$$

On the basis $F = e^{\omega y}$, it is seen that

$$\omega = \frac{2\rho}{b}(\pm p \pm iq)$$

where

$$\begin{aligned} p &= \sqrt{[(C_2-C_1)/2]} \\ q &= \sqrt{[(C_2+C_1)/2]}. \end{aligned} \tag{8}$$

The even eigenfunction can now be written as

$$F = \left[\frac{\cos(2q\rho y/b)}{\cos(q\rho)}\frac{\cosh(2p\rho y/b)}{\cosh(p\rho)}-\frac{\sin(2q\rho y/b)}{\sin(q\rho)}\frac{\sinh(2p\rho y/b)}{\sinh(p\rho)}\right] \tag{9.1}$$

where ρ is given by the transcendental equation

$$\frac{\sin(2q\rho)}{q}+\frac{\sinh(2p\rho)}{p} = 0. \tag{10.1}$$

The odd eigenfunction is

$$F = \left[\frac{\sin(2q\rho y/b)}{\sin(q\rho)}\frac{\cosh(2p\rho y/b)}{\cosh(p\rho)}-\frac{\cos(2q\rho y/b)}{\cos(q\rho)}\frac{\sinh(2p\rho y/b)}{\sinh(p\rho)}\right] \tag{9.2}$$

and the transcendental equation is

$$\frac{\sin(2q\rho)}{q}-\frac{\sinh(2p\rho)}{p} = 0. \tag{10.2}$$

The eigenfunctions for the isotropic case may be obtained as a special case from the above expressions by putting $E'_x = E'_y = E/(1-\mu^2)$, $E'' = \mu E/(1-\mu^2)$ and $G = E/2(1+\mu)$, so that $C_1 = C_2 = 1$, from which if follows $p = 0$, $q = 1$. In the limit of $p \to 0$ and $q = 1$, the expressions (9.1), (9.2), (10.1) and (10.2) reduce to the familiar Fadle–Papkovitch functions [1, 2].

Indeed, the condition $p = 0$, i.e. $C_1 = C_2$ is adequate to reduce the problem to an isotropic case. For, with the new variable $y_1 = y\sqrt{C_1}$, equations (5) and (6) assume the familiar forms of the isotropic problem (page 367, Timoshenko [10]).

In the above analysis it is implied that $C_2 > C_1$, which makes p a real quantity. At least for plywood materials, assuming the elastic constants given in Timoshenko [10], we have found p to be a real quantity. Even if p is imaginary, as it may happen for other materials, the above relations are valid.

The transcendental equations (10.1) and (10.2) have infinite number of complex roots expressible as $(\pm\alpha_K \pm i\beta_K)$, $K = 1, 2, \ldots$.

The asymptotic form for the Kth root of equation (10.1) (symmetric case) is found to be

$$\rho_K = \frac{1}{2(p^2+q^2)}\{[p\log(p/q)+(4K-1)\pi q/2]+i[(4K-1)\pi p/2-q\log(p/q)]\} \tag{11.1}$$

For the antisymmetric case, equation (10.2), it is given by

$$\rho_K = \frac{1}{2(p^2+q^2)}\{[p\log(p/q)+(4K+1)\pi q/2]+i[(4K+1)\pi p/2-q\log(p/q)]\} \tag{11.2}$$

Starting with the above expressions, the roots can be easily determined by Newton's iteration method in complex form, which is quite rapidly convergent. For the isotropic case, such a technique has been given by Buchwald [3].

In a similar manner, we develop another set of eigenfunctions $G_K(x)$ which satisfy the homogeneous conditions $G_K = \mathrm{d}G_K/\mathrm{d}x = 0$ at $x = \pm a/2$. Let

$$w = \cosh\left(\frac{2\rho' y}{a}\right)G(x).$$

On substituting this in equation (5), we obtain

$$\frac{\mathrm{d}^4 G}{\mathrm{d}x^4}+2C_1'\left(\frac{4\rho'^2}{a^2}\right)\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}+C_2'^2\left(\frac{16\rho'^4}{a^4}\right)G = 0$$

where

$$C_1' = H/D_x = (D_1+2D_{xy})/D_y = (E''+2G)/E_y'$$
$$C_2' = \sqrt{(D_y/D_x)} = \sqrt{(E_y'/E_x')}.$$

On the basis $G = \mathrm{e}^{\omega' x}$, it is found that

$$\omega' = \frac{2\rho'}{a}(\pm p' \pm iq')$$

where

$$p' = \sqrt{[(C_2'-C_1')/2]}$$
$$q' = \sqrt{[(C_2'+C_1')/2]}.$$

Now if

$$C_2 > C_1, \sqrt{(E_x'/E_y')} > (E''+2G)/E_y'$$

i.e.

$$\sqrt{(E'_x E'_y)} > E'' + 2G,$$

it follows $C'_2 > C'_1$.

Consequently the transcendental equations for ρ' are the same as before for ρ with p', q' replacing p, q respectively.† The function set $G_K(x)$ can be obtained by inserting p', q', ρ', a and x in the places of p, q, ρ, b and y respectively, in the expressions (9.1) and (9.2).

The even eigenfunction in the set $G_K(x)$ is

$$G = \left[\frac{\cos(2q'\rho' x/a)}{\cos(q'\rho')}\frac{\cosh(2p'\rho' x/a)}{\cosh(p'\rho')} - \frac{\sin(2q'\rho' x/a)}{\sin(q'\rho')}\frac{\sinh(2p'\rho' x/a)}{\sinh(p'\rho')}\right] \tag{12.1}$$

where ρ' is given by

$$\frac{\sin(2q'\rho')}{q'} + \frac{\sinh(2p'\rho')}{p'} = 0. \tag{13.1}$$

The odd eigenfunction is

$$G = \left[\frac{\sin(2q'\rho' x/a)}{\sin(q'\rho')}\frac{\cosh(2p'\rho' x/a)}{\cosh(p'\rho')} - \frac{\cos(2q'\rho' x/a)}{\cos(q'\rho')}\frac{\sinh(2p'\rho' x/a)}{\sinh(p'\rho')}\right] \tag{12.2}$$

and the transcendental equation is

$$\frac{\sin(2q'\rho')}{q'} - \frac{\sinh(2p'\rho')}{p'} = 0. \tag{13.2}$$

The infinite number of complex roots of the equations (13.1) and (13.2) are also expressible as $\pm\alpha'_K \pm i\beta'_K$, $K = 1, 2, \ldots$. The asymptotic forms for the kth root of ρ' is obtained by putting p', q' in the places of p and q in equations (11.1) and (11.2).

3. SOLUTION FOR ANY GIVEN LOADING

Having the two sets of eigenfunctions $F_K(y)$ and $G_K(x)$, a solution for the clamped orthotropic plate for any given loading $Q(x, y)$ can be developed on lines similar to that of Gaydon [6]. Assume that a particular integral w_0 of equation (2) can be found and write

$$w = w_0 + \Omega + w_1$$

where $\Omega(x, y)$ is a polynomial solution of the homogeneous equation (5) introduced for reasons discussed later. Then

$$D_x \frac{\partial^4 w_1}{\partial x^4} + 2H \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_1}{\partial y^4} = 0. \tag{14}$$

† The authors are indebted to Professor F. A. Gaydon for this proof.

The conditions (4.1) and (4.2) give at $x = \pm a/2$

$$\left.\begin{aligned} w_1 &= -(w_0+\Omega) \\ \frac{\partial w_1}{\partial x} &= -\frac{\partial}{\partial x}(w_0+\Omega) \\ \text{at } y &= \pm b/2 \\ w_1 &= -(w_0+\Omega) \\ \frac{\partial w_1}{\partial y} &= -\frac{\partial}{\partial y}(w_0+\Omega) \end{aligned}\right\}. \tag{15}$$

Now consider another function $w_2(x, y)$ which satisfies

$$D_x\frac{\partial^4 w_2}{\partial x^4}+2H\frac{\partial^4 w_2}{\partial x^2\partial y^2}+D_y\frac{\partial^4 w_2}{\partial y^4} = 0 \tag{16}$$

and the following conditions, at $x = \pm a/2$

$$\begin{aligned} \frac{\partial^2 w_2}{\partial y^2} &= -\frac{\partial^2}{\partial y^2}(w_0+\Omega) \\ \frac{\partial^2 w_2}{\partial x\partial y} &= -\frac{\partial^2}{\partial x\partial y}(w_0+\Omega) \\ \text{and at } y &= \pm b/2 \\ \frac{\partial^2 w_2}{\partial x^2} &= -\frac{\partial^2}{\partial x^2}(w_0+\Omega) \\ \frac{\partial^2 w_2}{\partial x\partial y} &= -\frac{\partial^2}{\partial x\partial y}(w_0+\Omega) \end{aligned} \tag{17}$$

It is not difficult to see that the relation between w_1 and w_2 is

$$w_1 = w_2+k_1x+k_2y+k_3 \tag{18}$$

where k_1, k_2 and k_3 are constants to be determined in any particular problem. If w_1 and w_2 are even in x, then $k_1 = 0$. If they are even in y, $k_2 = 0$ and if odd in either coordinate $k_3 = 0$. They are determined from known conditions, say, at corners.

The function w_2 can now be developed into a series of eigenfunctions $F_K(y)$, $G_K(x)$. Assuming that the loading is symmetric about both the axes, we write

$$w_2 = \sum_{\infty}^{\infty} A_K\cosh(2\rho_K x/b)F_K(y)+\sum_{\infty}^{\infty} B_K\cosh(2\rho'_K y/a)G_K(x) \tag{19}$$

where A_K, B_K are complex constants, $\rho_K \equiv \alpha_K+i\beta_K$, $\rho'_K \equiv \alpha'_K+i\beta'_K$ and A_{-K}, B_{-K}, ρ_{-K} etc., are the respective complex conjugates of A_K, B_K, ρ_K etc. Note that the sets of roots $-\alpha_K+i\beta_K$, $-\alpha'_K+i\beta'_K$ do not yield any new solutions. Applying the conditions (17) it will be seen that the problem is reduced to the determination of the complex constants A_K, B_K

satisfying the following equations:

$$x = \pm a/2 \begin{cases} \sum_{\infty}^{\infty} A_K \dfrac{d^2 F_K}{dy^2} = \dfrac{-\partial^2}{\partial y^2}(w_0 + \Omega) = f_1(y) & (20.1) \\ \dfrac{2}{b}\sum_{\infty}^{\infty} A_K \rho_K \dfrac{dF_K}{dy} = -\dfrac{\partial^2}{\partial x \partial y}(w_0 + \Omega) = f_2(y) & (20.2) \end{cases}$$

$$y = \pm b/2 \begin{cases} \sum_{\infty}^{\infty} B_K \dfrac{d^2 G_K}{dx^2} = \dfrac{-\partial^2}{\partial x^2}(w_0 + \Omega) = g_1(x) & (20.3) \\ \dfrac{2}{a}\sum_{\infty}^{\infty} B_K \rho'_K \dfrac{dG_K}{dx} = -\dfrac{\partial^2}{\partial x \partial y}(w_0 + \Omega) = g_2(x). & (20.4) \end{cases}$$

The purpose of the Ω will now be explained. The eigenfunctions F and G satisfy the following conditions:

$$\int_{-b/2}^{b/2} F \, . \, y \, dy = \int_{-b/2}^{b/2} \frac{dF}{dy} dy = \int_{-a/2}^{a/2} G \, . \, x \, . \, dx = \int_{-a/2}^{a/2} \frac{dG}{dx} dx = 0 \tag{21}$$

and the function set is complete with respect to these self-equilibrating conditions, though this is not proved here. Consequently for the expansions in equations (20.1) to (20.4) to be valid, we should impose that

$$\int_{-b/2}^{b/2} f_1 y \, dy = \int_{-b/2}^{b/2} f_2 \, dy = \int_{-a/2}^{a/2} g_1 x \, dx = \int_{-a/2}^{a/2} g_2 \, dx = 0. \tag{22.1}$$

Furthermore remembering the properties of the eigenfunctions viz.,

$$\frac{dF_K}{dy} = 0 \text{ at } y = \pm b/2, \quad \frac{dG_K}{dx} = 0 \text{ at } x = \pm a/2,$$

and in view of expansions (20.2) and (20.4), we require that

$$f_2|_{y = \pm b/2} = g_2|_{x = \pm a/2} = 0. \tag{22.2}$$

Fortunately all these requirements on the functions f_1, f_2, g_1 and g_2 can be met by a proper choice of Ω generated in polynomials forming solutions of (5).

4. RE-ORTHOGONALISATION OF EIGENFUNCTIONS

Unfortunately the functions $F_K(y)$, $G_K(x)$ do not seem to possess suitable orthogonality relations and indirect methods are to be employed for determining the constants in the expansions (20.1) to (20·4).

Although the usual methods such as least-squares etc., can be used, they do not possess special merits in the present problem. The techniques having certain advantages are (a) the method of biorthogonal functions developed by Johnson and Little [7], and Little [8], (*b*) the method of reorthogonalization developed by Gaydon and Shepherd [5], and Gaydon [6]. The latter method is adopted here. This consists of expanding each of the eigenfunctions into the orthogonal set of normalized clamped beam functions Y_m defined by:

$$\frac{d^4 Y_m}{dy^4} - (2\lambda_m/b)^4 Y_m = 0$$

under the conditions

$$Y_m = \frac{dY_m}{dy} = 0 \text{ at } y = \pm b/2.$$

In the symmetric case, the functions are

$$Y_m = \frac{1}{\sqrt{b}}\left[\frac{\cosh(2\lambda_m y/b)}{\cosh \lambda_m} - \frac{\cos(2\lambda_m y/b)}{\cos \lambda_m}\right]$$

where λ_m is given by $\tan \lambda_m + \tanh \lambda_m = 0$. The λ_ms are well-known [5]. Writing

$$F_K = \sum_{m=1}^{\infty} a_{Km} Y_m \tag{23}$$

by virtue of the orthogonality relation

$$\int_{-b/2}^{b/2} Y_m Y_n = \delta_{mn}$$

we find that

$$a_{Km} = \int_{-b/2}^{b/2} F_K \,.\, Y_m \,dy$$

The expression for a_{Km} is given in Appendix.

Also, it follows that

$$F_{-K} = \text{Conjugate of } F_K = \sum_{m=1}^{\infty} a_{-Km} Y_m$$

where a_{-Km} = Conjugate of a_{Km}.

Furthermore, because of the following orthogonality relations [5]

$$\int_{-b/2}^{b/2} \frac{d^2 Y_m}{dy^2} \frac{d^2 Y_n}{dy^2} dy = (2\lambda_m/b)^4 \delta_{mn}$$

$$\int_{-b/2}^{b/2} \frac{dY_m}{dy} \frac{d^3 Y_n}{dy^3} dy = -(2\lambda_m/b)^4 \delta_{mn}$$

the functions f_1, f_2 in equations (20.1) and (20.2) can be expanded in the following manner

$$f_1(y) = \sum_1^{\infty} c_m \frac{d^2 Y_m}{dy^2} \tag{23.1}$$

$$f_2(y) = \sum_1^{\infty} d_m \frac{dY_m}{dy} \tag{23.2}$$

where,

$$c_m = \left(\frac{b}{2\lambda_m}\right)^4 \int_{-b/2}^{b/2} f_1 \frac{d^2 Y_m}{dy^2} dy$$

$$d_m = -\left(\frac{b}{2\lambda_m}\right)^4 \int_{-b/2}^{b/2} f_2 \frac{d^3 Y_m}{dy^3} dy.$$

If we substitute the above expressions in equations (20.1) and (20.2), we obtain

$$\left.\begin{aligned} &\sum_{K=-\infty}^{\infty} A_K a_{Km} = c_m \\ &\frac{2}{b}\sum_{K=-\infty}^{\infty} A_K \rho_K a_{Km} = \mathrm{d}_m \end{aligned}\quad (m = 1, 2, \ldots)\right\} \tag{24}$$

which are the infinite system of equations required for the determination of A_K.

Similarly by expanding the set $G_K(x)$ into the orthogonal set $X_m(x)$ defined by

$$\frac{\mathrm{d}^4 X_m}{\mathrm{d}x^4} - (2\lambda_m/a)^4 X_m = 0, \qquad X_m = \frac{\mathrm{d}X_m}{\mathrm{d}x} = 0 \text{ at } x = \pm a/2$$

we obtain the following system of equations for B_K

$$\left.\begin{aligned} &\sum_{K=-\infty}^{\infty} B_K b_{Km} = c'_m \\ &\frac{2}{a}\sum_{K=-\infty}^{\infty} B_K \rho'_K \mathrm{b}_{Km} = \mathrm{d}'_m \end{aligned}\quad (m = 1, 2, \ldots)\right\}. \tag{25}$$

The expression for b_{Km} is obtained by putting a, ρ', p', q' in the places of b, ρ, p, q respectively in the expression for a_{Km} (Appendix).

The expressions for c'_m and d'_m are:

$$c'_m = \left(\frac{a}{2\lambda_m}\right)^4 \int_{-a/2}^{a/2} g_1 \frac{\mathrm{d}^2 X_m}{\mathrm{d}x^2}\mathrm{d}x$$

$$d'_m = -\left(\frac{a}{2\lambda_m}\right)^4 \int_{-a/2}^{a/2} g_2 \frac{\mathrm{d}^3 X_m}{\mathrm{d}x^3}\mathrm{d}x.$$

5. NUMERICAL EXAMPLE

As a numerical example, we consider the problem of a clamped plate under uniform normal pressure. A particular solution is

$$w_0 = \frac{1}{24}\frac{Q_0}{Dy}(b^2/4 - y^2)^2.$$

Since this automatically satisfies the conditions at $y = \pm b/2$, the function set $G_K(x)$ is not required. It is also seen that the function Ω is not required, as the functions $f_1 = -\partial^2 w_0/\partial y^2$; $f_2 = -\partial^2 w_0/\partial x \partial y$ satisfy the requirements (22.1) and (22.2). Equations (20.1) and (20.2) give

$$\sum_K A_K a_{Km} \frac{\cosh(\rho_K a/b)}{\cosh \rho_K} = c_m$$

$$\sum_K A_K a_{Km} \rho_K \frac{\sinh(\rho_K a/b)}{\cosh \rho_K} = 0$$

where

$$c_m = \frac{1}{8}\frac{Q_0}{Dy} b^4 b^{\frac{1}{2}} \frac{\tanh \lambda_m}{\lambda_m^5}.$$

These are the required infinite system of equations, which may be solved by the usual truncation procedure, using a computer. Since complex algebra can be handled on modern high speed computers, the separation of real and imaginary parts can be carried out on the computer itself.

For numerical work, we consider the plywood materials. The elastic constants are listed in Timoshenko [10] and Lechnitzky [11]. We reproduce these values in Table 1, and also present the values of p and q, defined by equation (8). Tables 2 to 6 give the eigenvalues, for symmetric case, for these materials. For a square plate ($a = b$), the numerical work has been carried out, by retaining only the first ten A_K and taking m up to 10. This means, each of the eigenfunctions is expanded into the first ten clamped beam functions resulting in a square matrix of size 20×20, for the ten unknown real parts, a_k, and the ten unknown imaginary parts, b_k, of the complex constants A_k. The diagonal terms of the matrix of the coefficients of a_k and b_k are found to be the largest, which justifies the truncation of the infinite system of equations. The coefficients γ_0 to γ_4 occurring in the following expressions for the deflection and moments are presented in Table 7.

$$w/_{\substack{x=0\\y=0}} = \gamma_0 Q_0 a^4/D_x$$

$$M_x/_{\substack{x=0\\y=0}} = \gamma_1 Q_0 a^2, \qquad M_y/_{\substack{x=0\\y=0}} = \gamma_2 Q_0 a^2$$

$$M_x/_{\substack{x=\pm a/2\\y=0}} = \gamma_3 Q_0 a^2, \qquad M_y/_{\substack{x=0\\y=\pm b/2}} = \gamma_4 Q_0 a^2.$$

TABLE 1. ELASTIC CONSTANTS FOR PLYWOOD IN BENDING
(x-axis is parallel to the face grain [9])

Material	Unit 10^6 psi					
	E'_x	E'_y	E''	G	p	q
Maple, 5-ply	1·87	0·60	0·073	0·159	0·746	1·100
Afara, 3-ply	1·96	0·165	0·043	0·110	0·964	1·588
Gabbon (Okoume), 3-ply	1·28	0·11	0·014	0·085	0·932	1·591
Birch, 3- and 5-ply	2·00	0·167	0·077	0·17	0·693	1·725
Birch with bakelite membranes	1·70	0·85	0·061	0·10	0·742	0·926

TABLE 2. EVEN EIGENVALUES FOR MAPLE 5-PLY

α_k	β_k
1·38189480	1·12401050
3·34143423	2·44260144
5·29769735	3·76932012
7·25395011	5·09601519
9·21020294	6·42271029
11·16645576	7·74940539
13·12270859	9·07610049
15·07896142	10·40279559
17·03521424	11·72949069
18·99146707	13·05618579

TABLE 3. EVEN EIGENVALUES FOR AFARA 3-PLY

α_k	β_k
1·00986320	0·77871471
2·46011174	1·65054511
3·90570665	2·52813040
5·35131412	3·40569048
6·79692157	4·28325067
8·24252902	5·16081086
9·68813647	6·03837105
11·13374392	6·91593124
12·57935137	7·79349143
14·02495882	8·67105162

TABLE 4. EVEN EIGENVALUES FOR GABOON (OKOUME) 3-PLY

α_k	β_k
1·02382621	0·77664605
2·49943556	1·63218568
3·96953534	2·49340025
5·43965707	3·35459019
6·90977871	4·21578024
8·37990035	5·07697029
9·85002198	5·93816034
11·32014362	6·79935039
12·79026526	7·66054043
14·26038690	8·52173048

TABLE 5. EVEN EIGENVALUES FOR BIRCH 3- AND 5-PLY

α_k	β_k
1·06875796	0·70045830
2·65281775	1·33025887
4·22091405	1·96003851
5·78903801	2·59001708
7·35716445	3·21999486
8·92529087	3·84997260
10·49341729	4·47995035
12·06154371	5·10992809
13·62967013	5·73990584
15·19779655	6·36988358

The numerical work has been carried out on CDC-3600. Further increase in the number of terms has shown no appreciable effect on the results. As a check on the accuracy of the solution, the deflection and the slopes at the centre of the clamped edges at $x = a/2$ are evaluated and found to be negligible. The ratio of the deflection at the edge to the central deflection is found to be less than $0{\cdot}7 \times 10^{-5}$ in all the cases.

Numerical results are also derived by applying a Galerkin method for a few cases; the results of the two methods agree with the limits of accuracy of the Galerkin method.

TABLE 6. EVEN EIGENVALUES FOR BIRCH WITH BAKELITE MEMBRANES

α_k	β_k
1·49197169	1·32261523
3·55724328	2·97003726
5·62333449	4·62556876
7·68940848	6·28110539
9·75548245	7·93664198
11·82155643	9·59217858
13·88763040	11·24771518
15·95370437	12·90325177
18·01977835	14·55878837
20·08585232	16·21432497

TABLE 7. DEFLECTION AND BENDING MOMENTS FOR A CLAMPED SQUARE PLATE UNDER UNIFORM LOAD

Material	γ_0	γ_1	γ_2	γ_3	γ_4
Maple, 5-ply	0·00023176	0·03691146	0·01595847	−0·07638084	−0·03274248
Afara, 3-ply	0·00001872	0·04431386	0·00877378	−0·08621326	−0·01638792
Gaboon (Okoume) 3-ply	0·00001976	0·04307108	0·00889494	−0·08569530	−0·01665159
Birch 3- and 5-ply	0·00001799	0·04636339	0·01409167	−0·08558040	−0·01645233
Birch with bakelite membranes	0·00052536	0·03321223	0·01864018	−0·07014675	−0·04132136

6. CONCLUSIONS

Eigenfunctions have been developed for the problem of clamped, rectangular, orthotropic plate undergoing small deflections. An analysis using these functions to obtain extremely accurate solutions is presented. The reasons for the accuracy in the solutions are that all the loads are made self-equilibrating and that it is the second derivatives that are matched at the boundary. These factors cannot be easily incorporated in the familiar methods such as multiple Fourier and energy methods. It is hoped that the present results provide a basis for comparison in the development of approximate techniques such as finite elements.

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APPENDIX

$$a_{Km} = \int_{-b/2}^{b/2} F_K Y_m \,\mathrm{d}y$$

$$= -16\sqrt{b}\frac{(q^2-p^2)}{\sin(2q\rho_K)}\frac{\lambda_m^2}{\rho_K^4}\frac{Z_1}{Z_2}$$

$$Z_1 = \{\lambda_m \tan \lambda_m \,.\, q(\cos 2q\rho_K + \cosh 2p\rho_K)/\sin(2q\rho_K) - \rho_K(p^2+q^2)\}$$

$$Z_2 = \{p^2+(q+\lambda_m/\rho_K)^2\}\{p^2+(q-\lambda_m/\rho_K)^2\}X\{q^2+(p+\lambda_m/\rho_K)^2\}\{q^2+(p-\lambda_m/\rho_K)^2\}.$$

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Абстракт—Разработаны собственные функции для защемленной, прямоугольной ортотропной пластинки, подверженной действию малых прогибов. Указано, что обыные функции Фадле-Паиковича (1) для случая изотропии образуют собой специальный случай более общих "ортотропных собственных функций". Анализ ортотропных пластинок, на основе зтих функций, дает чрезвичайно тщательные решения.